

Relations between the Ehrhart polynomial, the heat kernel and Sylvester waves

J.S.Dowker¹

*Theory Group,
School of Physics and Astronomy,
The University of Manchester,
Manchester, England*

I show for the specific case of the scalar field spectrum on regular tessellations of the sphere that the first two terms of the heat–kernel expansion are related to the first two terms of the Ehrhart (quasi)polynomial. In trying to make this relation precise, I consider degeneracies as partition denumerants and show the connection of the group theory expressions with Popoviciu’s theorem and with the notion of Sylvester waves. General denumerants are considered and the first wave, W_1 , *i.e.* the polynomial part, is written using the A –genus multiplicative sequence. It is pointed out that Sylvester in effect did the same thing and that he had also obtained Ehrhart reciprocity. I derive an algebraically neat form for the second wave, W_2 , which involves the combination of two multiplicative sequences.

¹dowker@man.ac.uk

1. Introduction

Connections between lattice and spectral problems go back over a hundred years. I mention only the Weyl conjecture and its physical antecedents (see, for example, Baltes and Hilf, [1]).

Two results, which are curiously similar, are the Ehrhart polynomial and the short-time expansion of the heat-kernel. Actually, rather than the heat-kernel, it is more appropriate to consider the asymptotic behaviour of a smoothed version, $\overline{N}(\lambda)$, of the exact eigenvalue counting function, $N(\lambda) = \sum_{k, \lambda_k \leq \lambda} 1$.² As a power series in λ , this has coefficients simply related to those of the heat-kernel expansion. For simplicity of exposition, I restrict initially to two dimensions and give the first two terms (which are all I am interested in) explicitly,

$$\overline{N}(\omega) \sim \frac{1}{4\pi} \left(|\mathcal{M}| \omega^2 \pm |\partial \mathcal{M}| \omega + c + \dots \right). \quad (1)$$

I have set λ equal to ω^2 and redefined $N(\omega) = N(\lambda)$.

These first two terms are those relevant for the Weyl conjecture, which states that (1) holds for the *exact* counting function. While this may not be true (actually it isn't for the manifolds I look at), (1) does hold for the smoothed version.

Turning to the lattice side, Ehrhart proved that the number of integer lattice points (*i.e.* elements of \mathbb{Z}^2) in, and on, a rational polygon, *i.e.* a polygon whose vertices are rational numbers, that has been uniformly dilated by an integer, l , is a *quasipolynomial* in l .

More precisely let \mathcal{P} be a rational polygon³ and $\overline{\mathcal{P}}$ its closure (*i.e.* including its boundary) and let $L(\overline{\mathcal{P}}, l) = \sharp(\overline{l\mathcal{P}} \cap \mathbb{Z}^2)$ be the number of integer points in the closure of the dilated polygon $l\mathcal{P} = \{(lx, ly) : (x, y) \in \mathcal{P}; l \in \mathbb{N}\}$. Then

$$L(\overline{\mathcal{P}}, l) = |\mathcal{P}| l^2 + c_1(l) l + c_0(l) \quad (2)$$

where c_1, c_0 are generally periodic functions in l (see, for example, Beck and Robins, [2] and Wright [3]).

In the special case that the polygon is an *integer* polygon the coefficients c_1, c_0 are constant and, moreover,

$$L(\overline{\mathcal{P}}, l) = |\mathcal{P}| l^2 + \frac{1}{2} |\partial \mathcal{P}| l + c_0 \quad (3)$$

² The eigenvalues, λ_k are ordered linearly, with k a counting label.

³I sometimes refer to a disc with polygonal boundary (a 2-polytope) as a polygon, for short.

which bears a remarkable similarity to (1).

The two expressions cannot be related in all circumstances because not all lattice counting problems have spectral associations. This paper, however, is concerned with a very particular case in which they can be more precisely connected.

In sections 2 and 3, I set up the situation and present an experimental result. The later sections are intended to be more exact and give some explanation of the result. This paper thus proceeds from the particular to the general.

2. The factored sphere

The situation I refer to is very well known and so I need not spend time on the underlying mathematical details. They have been dealt with by, for example, Gromes, [4], Bérard and Besson, [5], Brüning and Heintze, [6] and Chang and Dowker, [7]. The manifold, \mathcal{M} is an orbifold factor of the two-sphere, S^2/Γ , where Γ is the reflective symmetry group of one of the regular solids, a finite subgroup of $O(3)$. \mathcal{M} is therefore a spherical triangle tiling the sphere, or, in the dihedral case, it is a lune (the situation discussed by Gromes).

Making the special choice of conformal coupling in $1 + 2$ dimensions means that the eigenvalues of the corresponding ‘improved’ Laplacian are perfect squares of integers, or half-odd integers, (for the unit sphere). By an appropriate selection of the action of the tiling group, Γ , one can arrange Dirichlet or Neumann conditions on the boundary of the fundamental domain, \mathcal{M} . A calculation, by separation of variables if you wish, *e.g.* [4] on the lune, gives the eigenvalues, $\lambda_{\mathbf{m}}$, determined by

$$\omega_{\mathbf{m}} \equiv \sqrt{\lambda_{\mathbf{m}}} = a + d_1 m_1 + d_2 m_2, \quad m_1, m_2 = 0, 1, 2, \dots \quad (4)$$

where d_1 and d_2 are the integer *degrees* (not necessarily coprime) associated with the action of Γ , *i.e.* with the particular regular solid (3-polytope). For example, for the lune of apex angle π/q ($q \in \mathbb{Z}$), $d_1 = q$, $d_2 = 1$. The hemisphere corresponds to $q = 1$.

The constant, a , takes the values

$$a_N = \frac{1}{2} \quad \text{and} \quad a_D = d_1 + d_2 - a_N$$

for Neumann and Dirichlet conditions, [7].

One now sees the relation with Ehrhart polynomials. Trivially, counting eigenvalues amounts to counting integer lattice points. It is no surprise that similar quantities appear.

Explicitly, consider the (special) rational triangle (2-polytope)

$$\mathcal{P}_{\mathbf{d}} = \left\{ (x_1, x_2) \in \mathbb{R}^2 : x_i \geq 0, d_1 x_1 + d_2 x_2 \leq 1 \right\}$$

with vertices $(0, 0)$, $(1/d_1, 0)$ and $(0, 1/d_2)$ where d_1 and d_2 are relatively prime, which is not an essential restriction. Then construct the integer dilation, $l\mathcal{P}_{\mathbf{d}}$, and count the number, $L(\overline{\mathcal{P}}_{\mathbf{d}}, l)$ of integer lattice points, (m_1, m_2) , inside (including the boundary as described previously). The result is given by (2) where now in fact c_1 is a constant (because d_1 and d_2 are coprime). Hence, so far,

$$L(\overline{\mathcal{P}}_{\mathbf{d}}, l) = \frac{l^2}{2d_1 d_2} + c_1 l + c_0(l).$$

The complete expression has been obtained by Beck and Robins, [2], and I reproduce the essential parts here,

$$L(\overline{\mathcal{P}}_{\mathbf{d}}, l) = \frac{1}{2d_1 d_2} \left(l^2 + l(d_1 + d_2 + 1) + \frac{1}{6}(d_1^2 + d_2^2 + 3d_1 d_2 + 1) + \sigma(l) \right). \quad (5)$$

where $\sigma(l)$ is a periodic function involving Dedekind sums. The origin, Ehrhart [8], should also be consulted.

To make a connection with eigenvalues, the essential counting restriction is

$$d_1 m_1 + d_2 m_2 \leq l$$

which, in terms of eigenvalues, (4), reads

$$\omega_{\mathbf{m}} \leq l + a.$$

so that the identification

$$N(\omega) = L(\overline{\mathcal{P}}_{\mathbf{d}}, \omega - a) \quad (6)$$

can be made *but* only when $\omega - a$ is an integer. As a function of a real l , $L(\overline{\mathcal{P}}_{\mathbf{d}}, l)$ will provide an interpolation and then the corresponding interpolated $N(\omega)$ from (6) would give a smoothing of the exact $N(\omega)$. To see how this compares with the smoothed expression (1), I put in the geometric values appropriate for a fundamental domain on the 2-sphere to get for (1),

$$\overline{N}(\omega) \sim \frac{1}{2d_1 d_2} \left(\omega^2 \pm (d_1 + d_2 - 1) \omega + c' + \dots \right). \quad (7)$$

The factor $2d_1d_2$ is the order of the reflective tiling group, Γ and the d_1 and d_2 need not be coprime.

Choosing the upper (Neumann) sign for ease ($a = 1/2$), substitution of $l = \omega - 1/2$ in (5) does not yield (7). However, all is not lost. Plotting the exact lattice counting expression (5) at integer values of l produces a familiar staircase function. The interpolation provided by (5) for real l joins the upper part of the steps. The curve that joins the lower part is given by $L(\overline{\mathcal{P}}_{\mathbf{d}}, l - 1)$ and averaging these produces an interpolation (*i.e.* smoothing) that passes half way up the vertical rises. It is easily confirmed that this symmetrical combination

$$\frac{1}{2}(L(\overline{\mathcal{P}}_{\mathbf{d}}, \omega - 1/2) + L(\overline{\mathcal{P}}_{\mathbf{d}}, \omega - 3/2)) \quad (8)$$

reproduces the first two terms of (7), with the upper sign. This is the conclusion so far.

Also one obtains the equality

$$|\mathcal{M}| = 4\pi|\mathcal{P}_{\mathbf{d}}|.$$

3. Higher dimensions

To show that this result possibly goes beyond mere numerical coincidence (which is unlikely anyway in view of the dependence on d_1 and d_2), I look at three dimensions more particularly.

Generally, in d dimensions, the measures of the fundamental domain and its boundary are given by

$$|\mathcal{M}| = \frac{1}{2g}|S^d| \quad \text{and} \quad |\partial\mathcal{M}| = \frac{b_1}{g}|S^{d-1}|$$

in terms of the order, $2g$, of the tiling group, Γ , and the number, b_1 , of reflecting great hyperspheres (or reflecting d -flats in the embedding $(d+1)$ -dimensional Euclidean manifold). In terms of the degrees

$$g = \prod_{i=1}^d d_i \quad \text{and} \quad b_1 = \sum_{i=1}^d (d_i - 1) + 1$$

so that, in particular for $d = 3$, $g = d_1d_2d_3$ and $b_1 = d_1 + d_2 + d_3 - 2$.

Writing out the conventional asymptotic behaviour of the smoothed counting function in this particular situation gives

$$\overline{N}(\omega) \sim \frac{1}{2g\Gamma(d)} \left(\frac{2}{d} \omega^d \pm b_1 \omega^{d-1} + \frac{d-1}{6} (b_1(b_1-1) + b_2) \omega^{d-2} + \dots \right). \quad (9)$$

where I have included the next term which involves b_2 , the number of elements of the group Γ that fix a $(d-1)$ -flat but not a d -flat. In three dimensions,

$$b_2 = d_2 d_3 + d_3 d_1 + d_1 d_2 - d_1 - d_2 - d_3.$$

In three dimensions the Neumann constant, a_N , equals 1 and the suggested symmetrical (midway) combination is, instead of (8),

$$\frac{1}{2} (L(\overline{\mathcal{P}}_{\mathbf{d}}, \omega - 1) + L(\overline{\mathcal{P}}_{\mathbf{d}}, \omega - 2)). \quad (10)$$

The relevant polynomial terms have been determined by Beck *et al*, [9], and again I write them out

$$L(\overline{\mathcal{P}}_{\mathbf{d}}, l) = \frac{1}{2d_1 d_2 d_3} \left(\frac{l^3}{3} + \frac{l^2}{2} (d_1 + d_2 + d_3 + 1) + \frac{l}{6} (3(d_1 + d_2 + d_3 + d_2 d_3 + d_3 d_1 + d_1 d_2) + d_1^2 + d_2^2 + d_3^2) + \dots \right), \quad (11)$$

where the omitted terms are periodic.

Evaluating (8) yields the expression

$$\frac{1}{2d_1 d_2 d_3} \left(\frac{\omega^3}{3} + b_1 \frac{\omega^2}{2} + \frac{(b_1(b_1-1) + b_2 + 1)\omega}{6} + \dots \right)$$

the first two terms of which agree with the ‘standard’ expansion (9). The third term differs, but only very slightly, which proximity I cannot explain. One might expect that one should reproduce those polynomial coefficients which are constant, *i.e.* all those except the final one, proportional to ω^0 , although the higher heat-kernel coefficients depend on the propagation equation.

4. Degeneracies as denumerants

I now wish to proceed to some exact relations, with the aim of making the previous conjecture more sensible. They are also more interesting.

Averaging the counting function involves loss of information. I return to the exact function which determines, and is determined by, the (square root) eigenvalue spectrum, $\{\omega_k\}$, $k = 1, 2, \dots$, coincidences implying degeneracy.

The spectrum can also be specified by the distinct *eigenlevels*, $\omega(l)$ say, $l = 0, 1, 2, \dots$, and the corresponding degeneracies, $g(l)$. On the fundamental domain, leaving the eigenvalues in the form (4) expressed in terms of two integers (or more for higher spheres) one can often avoid explicitly constructing the degeneracies. However degeneracies have a useful role and appear in other contexts.

For a given Neumann eigenlevel, $\omega(l)$, the number of integer lattice points (m_1, m_2) satisfying

$$d_1 m_1 + d_2 m_2 = l = \omega(l) - a, \quad \text{with} \quad a = \frac{1}{2}, \quad (12)$$

gives the degeneracy, *i.e.*

$$g_N(l; \mathbf{d}) = \# \left((m_1, m_2) \in \mathbb{Z}_+^2 : d_1 m_1 + d_2 m_2 = l \right).$$

$g_N(l; \mathbf{d})$ counts the number of lattice points on the hypotenuse of the dilated rational triangle $l\mathcal{P}_{\mathbf{d}}$ and is often referred to as a restricted partition or a denumerant, in an older terminology. Bell, [10], denotes the obvious extension to higher dimensions by $D(l \mid \mathbf{d})$ and Sylvester employs a variety of notations and terminology. His final one seems to be

$$D(l \mid \mathbf{d}) = \frac{l}{d_1, d_2, \dots, d_d},$$

which I shorten to l/\mathbf{d} .⁴

The study of denumerants is effectively the same as that of Ehrhart (quasi) polynomials as discussed later. Ehrhart himself, [8], actually spends most analytical time on them.

The associated generating function is well known, going back at least to Euler, [11],

$$h_N(\sigma; \mathbf{d}) \equiv \sum_{l=0}^{\infty} g_N(l; \mathbf{d}) \sigma^l = \frac{1}{(1 - \sigma^{d_1})(1 - \sigma^{d_2})}. \quad (13)$$

⁴ I retain the geometric terminology of ‘degrees’ for the denominator of the denumerant. Sylvester refers to them as the ‘components’ of the denumerant, among other things. Another possibility would be ‘parameters’.

Of course, as a lattice statement, this holds for any integers d_1 and d_2 . For the S^2/Γ spectral problem, d_1 and d_2 take only certain values and the generating function (13) is just the Molien series for a finite reflection group action, *e.g.* Stanley, [12], Meyer [13] and was explored by Laporte, [14], from a physical, mode point of view. The extension to higher dimensions is immediate. See Brüning and Heintze, [6], Bérard and Besson, [5].

Dirichlet conditions correspond to a different value for a in (12) and amount to a shift in l by a constant d_0 where

$$d_0 = d_1 + d_2 - 1,$$

and is the number of reflecting hyperplanes in the embedding space.

Then the Dirichlet generating function is

$$h_D(\sigma; \mathbf{d}) \equiv \sum_{l=0}^{\infty} g_D(l; \mathbf{d}) \sigma^l = \frac{\sigma^{d_0}}{(1 - \sigma^{d_1})(1 - \sigma^{d_2})}, \quad (14)$$

or

$$g_D(l + d_0; \mathbf{d}) = g_N(l; \mathbf{d}), \quad l = 0, 1, 2, \dots \quad (15)$$

As an important example, for the lune of angle π/q , $q \in \mathbb{Z}$,

$$\begin{aligned} h_N(\sigma; q, 1) &= \frac{1}{(1 - \sigma^q)(1 - \sigma)} \\ h_D(\sigma; q, 1) &= \frac{\sigma^q}{(1 - \sigma^q)(1 - \sigma)} \\ &= h_N(\sigma; q, 1) - \frac{1}{1 - \sigma}. \end{aligned}$$

A classic roots of unity, or trigonometric, calculation gives the explicit values for the degeneracies

$$g_N(l; q, 1) = \lfloor l/q \rfloor + 1 = \lceil l/q \rceil, \quad g_D(l; q, 1) = \lfloor l/q \rfloor,$$

the solutions for the corresponding denumerants, (12), *e.g.* Sylvester [15].

Combining the Dirichlet and Neumann spectra (which amounts to adding spectral quantities such as the generating functions) gives the spectrum for a doubled fundamental domain on which the modes are periodic under the pure rotational part of Γ . Thus for the periodic lune (of angle $2\pi/q$) we get the standard formula,

$$\begin{aligned} h(\sigma; q, 1) &\equiv h_N(\sigma; q, 1) + h_D(\sigma; q, 1) = \frac{1 + \sigma^q}{(1 - \sigma^q)(1 - \sigma)} \\ &= \sum_{t=0}^{\infty} (2\lfloor t/q \rfloor + 1) \sigma^t, \end{aligned} \quad (16)$$

which exhibits the usual degeneracies sometimes obtained from group characters, *e.g.* [7], Harmer, [16].

For the finite number of periodic, uniform sphere tilings, the generating functions can be obtained by combining cyclic expressions corresponding to a geometric decomposition of the action of the subgroup of $\text{SO}(3)$ in terms of axes and orders, [13,17,7]. Using the orbit–stabiliser theorem, this leads to the elegant result, [18],

$$h(\sigma; \mathbf{d}) = \frac{1}{2} \left(\sum_q h(\sigma; q, 1) - h(\sigma; 1, 1) \right)$$

where the sum is over all axes of order q . As an example, the tetrahedral group has $d_1 = 3$, $d_2 = 4$ and $q = 2, 3, 3$. Simple algebra yields, using just knowledge of the qs ,

$$h(\sigma; 3, 4) = \frac{1 + \sigma^6}{(1 - \sigma^3)(1 - \sigma^4)},$$

from which the values of d_1 and d_2 could be read off, if they weren't known.

Using the cyclic degeneracies, (16), one derives the explicit expression for the rotational tetrahedral degeneracies,

$$\begin{aligned} g(l; 3, 4) &= \left\lfloor \frac{l}{2} \right\rfloor + 2 \left\lfloor \frac{l}{3} \right\rfloor + 1 - l \\ &= \frac{l}{6} + 1 - \left\{ \frac{l}{2} \right\} - 2 \left\{ \frac{l}{3} \right\}. \end{aligned} \tag{17}$$

For comparison, I reproduce the corresponding formulae for the octahedral and icosahedral tilings, [18],

$$\begin{aligned} g(l; 4, 6) &= \left\lfloor \frac{l}{2} \right\rfloor + \left\lfloor \frac{l}{3} \right\rfloor + \left\lfloor \frac{l}{4} \right\rfloor + 1 - l \\ &= \frac{l}{12} + 1 - \left\{ \frac{l}{2} \right\} - \left\{ \frac{l}{3} \right\} - \left\{ \frac{l}{4} \right\}, \end{aligned} \tag{18}$$

and

$$\begin{aligned} g(l; 6, 10) &= \left\lfloor \frac{l}{2} \right\rfloor + \left\lfloor \frac{l}{3} \right\rfloor + \left\lfloor \frac{l}{5} \right\rfloor + 1 - l \\ &= \frac{l}{30} + 1 - \left\{ \frac{l}{2} \right\} - \left\{ \frac{l}{3} \right\} - \left\{ \frac{l}{5} \right\}. \end{aligned} \tag{19}$$

Returning now to strictly lattice considerations, the solution of (13) for the lattice quantity, $g_N(t; \mathbf{d})$, with any coprime d_1 and d_2 , is given by Popoviciu's theorem,

$$g_N(l; \mathbf{d}) = \frac{l}{d_1 d_2} + 1 - \left\{ \frac{d_1^{-1} l}{d_2} \right\} - \left\{ \frac{d_2^{-1} l}{d_1} \right\}, \tag{20}$$

where d_2^{-1} is the $\text{mod}(d_1)$ inverse of d_2 and d_1^{-1} that of d_1 , $\text{mod}(d_2)$.

I wish to apply this to the tetrahedral case to check (17). I note that $3^{-1}|_{\text{mod}4} = 3$ and $4^{-1}|_{\text{mod}3} = 1$ so that (20) reads

$$\begin{aligned} g_N(l; 3, 4) &= \frac{l}{12} + 1 - \left\{ \frac{3l}{4} \right\} - \left\{ \frac{l}{3} \right\} \\ &= \left[\frac{3l}{4} \right] + \left[\frac{l}{3} \right] + 1 - l. \end{aligned} \quad (21)$$

This is shown to be consistent with the rotational tetrahedral degeneracies, (17), by constructing the combination

$$g_N(l; 3, 4) + g_N(l - 6; 3, 4) = \frac{l}{6} + \frac{3}{2} - \left\{ \frac{3l}{4} + \frac{1}{2} \right\} - \left\{ \frac{3l}{4} \right\} - 2 \left\{ \frac{l}{3} \right\}$$

according to the relation (15). Equivalence results in view of the equality,

$$\left\{ \frac{3l}{4} + \frac{1}{2} \right\} + \left\{ \frac{3l}{4} \right\} = \left\{ \frac{l}{2} \right\} + \frac{1}{2},$$

essentially just a Hermite identity.

This result means that one has explicit, *separate* formulae for the Neumann and Dirichlet mode numbers, discussed case by case by Laporte, [14]. I now check the other tilings,

For this I require an extension of Popoviciu's theorem which is, (see Beck and Robins, [2] ex.1.28) that, if $\text{gcd}(d_1, d_2) = e$, then

$$\begin{aligned} g_N(l; \mathbf{d}) &= \frac{le}{d_1 d_2} - \left\{ \frac{\delta_1 l}{d_2} \right\} - \left\{ \frac{\delta_2 l}{d_1} \right\} + 1, \quad \text{if } e|l \\ &= 0, \quad \text{otherwise.} \end{aligned} \quad (22)$$

where $\delta_1 d_1/e = 1 \text{ mod } (d_2/e)$ and $\delta_2 d_2/e = 1 \text{ mod } (d_1/e)$.

In the octahedral and icosahedral cases, the greatest common divisor is 2 and l must be even for non-zero g_N in (22). In the octahedral case, the reduced mod inverses are $\delta_1 = 2$ and $\delta_2 = 1$ so that

$$g_N(l; 4, 6) = \frac{l}{12} + 1 - \left\{ \frac{l}{4} \right\} - \left\{ \frac{l}{3} \right\}, \quad l \text{ even} \quad (23)$$

which agrees with (18) because $\{l/2\}$ is zero for even l .

The Dirichlet modes are obtained by applying (15) to give

$$\begin{aligned}
g_D(l; 4, 6) &= \frac{l-9}{12} + 1 - \left\{ \frac{l-9}{4} \right\} - \left\{ \frac{l-9}{3} \right\}, \quad l \text{ odd} \\
&= \frac{l}{12} + \frac{1}{4} - \left\{ \frac{l-1}{4} \right\} - \left\{ \frac{l}{3} \right\} \\
&= \frac{l}{12} + \frac{1}{2} - \left\{ \frac{l}{4} \right\} - \left\{ \frac{l}{3} \right\}
\end{aligned} \tag{24}$$

again agreeing with (18), for l odd.

A similar calculation holds for the icosahedral case where $\delta_1 = 2$, $\delta_2 = 2$. Substitution into (22) immediately yields equality with the icosahedral degeneracies, (19).

In these two cases the rotational formula gives both the N and D values because these are associated with even and odd l respectively, whilst they are mixed up for the tetrahedron. A similar cross linking complication arises when computing spectral quantities for twisted vector bundles over homogeneous factors of the three-sphere, [19].

For the tetrahedron, because the degrees are coprime, there exists a Frobenius number which equals $d_1 d_2 - d_1 - d_2 = 5$, meaning that there is at least one N -mode for every integer l greater than 5. This is not true for the octahedron or icosahedron as the degrees are both even.

5. Sylvester waves

It is seen that all the relevant quantities take the standard form of a polynomial in l plus a periodic term, in agreement with an ancient partition theorem of Sylvester's, first stated in [15] to the effect that the denominator takes the form,

$$\frac{l}{\mathbf{d}} = P(l) + U, \tag{25}$$

where $P(l)$ is a polynomial in l of degree $d-1$ (I work in d dimensions) and U , the 'undulant' part, is periodic and contains roots of unity.

Because of this periodicity, Sylvester refers to the parts of l/\mathbf{d} , in (25) as 'waves', depending on all roots of unity (most waves vanish by natural selection).

For example, in the rotational tetrahedral degeneracy, (17), the periodic part has period 6 (the individual N and D parts having period 12). The leading term of the polynomial part is classic, [20].

Sylvester, [15,21], gives the rule for calculating the polynomial part, P , as the coefficient of $1/t$ in the expansion of the quantity

$$\frac{e^{lt}}{\prod_{i=1}^d (1 - e^{-d_i t})}, \quad (26)$$

which is clearly a polynomial in l .

By definition, this is directly expressed as a generalised Bernoulli function,

$$\begin{aligned} P_d &= \frac{1}{(d-1)! \prod_i d_i} B_{d-1}^{(d)}(l + \sum_i d_i \mid \mathbf{d}) \\ &= \frac{(-1)^{d-1}}{(d-1)! \prod_i d_i} B_{d-1}^{(d)}(-l \mid \mathbf{d}) \end{aligned} \quad (27)$$

which is a polynomial in l whose coefficients are generalised Bernoulli numbers (up to a factor).

This expression for the polynomial part of the denominator has been noted by Rubinstein and Fel, [22] who also point out the equivalence with a direct evaluation by Beck, Gessel and Komatsu, [23] which seems more or less the same as Sylvester's, [21].

The representation (27) is, in some ways, only cosmetically compact. It has value when general properties are being investigated and recursion relations *etc.* can be brought in. These can also be used to find explicit forms for particular cases, as discussed by Norlund, [24] and the Bernoulli polynomials can be computed in various ways. However, I prefer here to use the relation, employed in [25], with Todd polynomials, T_n , [26],

$$B_r^{(d)}(x \mid \mathbf{d}) = (-1)^r r! \sum_{s=0}^r (-1)^s \frac{x^s}{s!} T_{r-s}(\sigma_1, \dots, \sigma_{r-s}), \quad r \leq d, \quad (28)$$

where the σ_s are symmetric functions of the degrees d_i ($i = 1, \dots, d$). This form has the advantage of being valid for all d . That is, the *functional form* in the σ_i of the right-hand side does not depend on d . Accordingly, the expansion reads,

$$\begin{aligned}
P_d &= \frac{(-1)^{d-1}}{(d-1)! \prod_i d_i} B_{d-1}^{(d)}(-l \mid \mathbf{d}) \\
&= \frac{(-1)^{d-1}}{(d-1)! \prod_i d_i} \sum_{k=0}^{d-1} \binom{d-1}{k} (-l)^k B_{d-k-1}^{(d)}[\mathbf{d}] \\
&= \frac{1}{\prod_i d_i} \sum_{k=0}^{d-1} \frac{l^k}{k!} T_{d-1-k}(\sigma_1, \dots, \sigma_{d-1-k}) \\
&= \frac{1}{\prod_i d_i} \left(\frac{l^{d-1}}{(d-1)!} T_0 + \frac{l^{d-2}}{(d-2)!} T_1 + \frac{l^{d-3}}{(d-3)!} T_2 + \dots \right)
\end{aligned} \tag{29}$$

Known forms for the Todd polynomials make the coefficients explicit, which allows checking, but the real advantage of using this structure is that the Todd polynomials are independent of d and one sees that the expression for dimension d is obtained by integrating that for $(d-1)$, the only ‘new’ term being the constant. I now present another (equivalent) version of this expansion.

From an algebraic viewpoint a more advantageous variable is $\bar{l} = l + \frac{1}{2} \sum_i d_i$. Sylvester, [27], calls this the *augmented* argument and also extends l to a real number so allowing the use of analysis, as, in fact, I have just done.

A basic property of the Bernoulli functions, [24,28], shows that, *expressed as a function of \bar{l}* , the polynomial part has the parity property,

$$P_d(-\bar{l}) = (-1)^{d-1} P_d(\bar{l}). \tag{30}$$

Furthermore, the generating function equation takes on a more symmetrical aspect in terms of \bar{l} ,

$$\sum_{l=0}^{\infty} \frac{l}{\mathbf{d},} \sigma^{\bar{l}} = \frac{1}{\prod_i (\sigma^{-d_i/2} - \sigma^{d_i/2})},$$

which shows that, as a function of \bar{l} , the full denominator also obeys the parity property (Ehrhart reciprocity). It follows that the undulant part satisfies it also. This was proved by Ehrhart geometrically who thence *derived* the parity behaviour of the polynomial part. It seems that Sylvester, [27], had derived full reciprocity somewhat earlier and would also have had the reciprocity of Dedekind sums, if he had defined them.

It is instructive to make the expansion (29) reflect the parity more easily. It

can be rewritten (using [24] p.162, (18)),

$$\begin{aligned} P_d &= \frac{1}{(d-1)! \prod_i d_i} B_{d-1}^{(d)}(\bar{l} + \sum d_i/2 \mid \mathbf{d}) \\ &= \frac{1}{2^{d-1}(d-1)! \prod_i d_i} \sum_{k=0}^{d-1} \binom{d-1}{k} (2\bar{l})^k D_{d-k-1}^{(d)}[\mathbf{d}] \end{aligned} \quad (31)$$

in terms of the computable constants, $D_\nu^{(d)}[\mathbf{d}]$ which vanish for odd ν . (This is the statement of reciprocity for this term).

Again, the constants $D_{2\nu}^{(d)}[\mathbf{d}]$ are essentially the A_ν polynomials, [26], and Hirzebruch helpfully gives this time, the relation,

$$D_{2\nu}^{(d)}[\mathbf{d}] = \frac{(2\nu)!}{2^{2\nu}} A_\nu(p_1, \dots, p_\nu), \quad 2\nu \leq d, \quad (32)$$

and the \bar{l} expansion is,

$$P_d = \frac{1}{\prod_i d_i} \left(\frac{\bar{l}^{d-1}}{(d-1)!} A_0 + \frac{1}{16} \frac{\bar{l}^{d-3}}{(d-3)!} A_1 + \frac{1}{2^8} \frac{\bar{l}^{d-5}}{(d-5)!} A_2 + \dots \right), \quad (33)$$

which terminates.

Hirzebruch lists a few of the A polynomials as functions of the indeterminates, p_i , which are, here, the elementary symmetric functions of the *squares* of the degrees, [26], §1.3 which, to remind, are d in number. Thus,

$$A_1 = -\frac{2}{3}p_1, \quad A_2 = \frac{2}{45}(7p_1^2 - 4p_2), \quad A_3 = -\frac{4}{3^3 \cdot 5 \cdot 7}(16p_3 - 44p_2p_1 + 31p_1^3). \quad (34)$$

I also reproduce some values of the B and D constants as given in [24], p.167.

$$\begin{aligned} B_0^{(n)} &= 1, & B_1^{(n)} &= -\frac{1}{2}\sigma_1, & B_2^{(n)} &= \frac{1}{6}s_2 + \frac{1}{2}\sigma_2 \\ D_0^{(n)} &= 1, & D_2^{(n)} &= -\frac{1}{3}s_2, & D_4^{(n)} &= \frac{7}{15}s_4 + \frac{2}{3} \sum_{i < j} d_i^2 d_j^2 \end{aligned} \quad (35)$$

where the s_p are the sums of the p -th powers of the degrees, d_i . These expressions are easily shown to agree with those obtained via the Todd or the A -polynomials. Furthermore, they hold for all n and the conclusion is that one can relax the conditions in (28) and (32). Thus the range of 2ν in (32) can be extended beyond d and, for example, from (34), one gets $D_4^{(1)} = 7d_1^4/15$ agreeing with (35).⁵

⁵ More generally, the coefficient of the term p_1^ν in A_ν equals $(2^{2\nu}/(2\nu)!) D_{2\nu}$ where the $D_{2\nu}$ are listed in Table 4 in [28] and are the coefficients in the expansion of the characteristic series $x/\sinh x$.

It should be apparent that there is no need to introduce generalised Bernoulli polynomials. It is possible to move to the final formulae in (29) and (33) directly in terms of multiplicative sequences. In fact this is what Sylvester does in his explicit determinations.

The polynomial part, P is that Sylvester wave corresponding to the root of unity, 1. He denotes it by W_1 , and in his series of papers on partitions explains, several times, a systematic computational scheme for its algebraic determination, the most detailed being [27]. There he obtains, by rapid direct calculation, an expansion identical to (33), except that the polynomial coefficients are essentially all the homogenous products (of a given order) expressed in terms of functions of the sums of even *powers* of the degrees. I review the calculational details. A good introduction to Sylvester's theory can be found in the textbook by Netto, [29].

Instead of (26), one now has the more symmetrical form,

$$W_1 = \text{co}_{-1} \frac{e^{\bar{l}t}}{\prod_{i=1}^d (e^{d_i t/2} - e^{-d_i t/2})} = \text{co}_{-1} \frac{e^{\bar{l}t}}{\prod_{i=1}^d 2 \sinh \frac{1}{2} t d_i}. \quad (36)$$

Expansion of the numerator yields the coefficient of the power $\bar{l}^n/n!$, as

$$\begin{aligned} \text{co}_{-1} \frac{t^n}{\prod_i 2 \sinh \frac{1}{2} t d_i} &= \frac{1}{\prod d_i} \text{co}_{-1} t^{n-d} \prod_i \frac{x_i}{\sinh x_i} \\ &= \frac{1}{\prod d_i} \text{co}_{d-1-n} \prod_i \frac{x_i}{\sinh x_i} \end{aligned} \quad (37)$$

where I have set $x_i = t d_i/2$ and have encountered a multiplicative sequence, K , with characteristic function $Q(x) = x/\sinh x$ so that

$$Q(x_1)Q(x_2)\dots = 1 + K_2(s_1, s_2) + K_4(s_1, s_2, s_3, s_4) + \dots \quad (38)$$

where s_k is the sum of the k th powers of the x_i . It is more conventional to use the symmetrical products, but this choice is more natural, as will be seen. The K_m are homogeneous of degree m in the x_i and so, in view of the definition of the x_i , the right-hand side of (38) is a power series in t^2 . According to (37) the required coefficient is simply K_{d-1-n} . To actually compute this, it is necessary to expand the particular characteristic function as a power series. Although everything is classical, I will do this *ab initio* from the old product

$$Q(x) = \frac{x}{\sinh x} = \prod_{n=1}^{\infty} \frac{1}{(1 + x^2/n^2\pi^2)}. \quad (39)$$

Therefore

$$\begin{aligned}\log Q(x) &= - \sum_{n=1}^{\infty} \log(1 + x^2/n^2\pi^2) \\ &= -\frac{\zeta(2)}{\pi^2} x^2 + \frac{\zeta(4)}{2\pi^4} x^4 - \dots + (-1)^k \frac{\zeta(2k)}{k\pi^{2k}} x^{2k} \pm \dots\end{aligned}\quad (40)$$

The expansion of $\log Q$ rather than Q is the tactical point here and the construction of the multiplicative sequence, (38), follows on exponentiation,

$$\begin{aligned}Q_1 Q_2 \dots &= \exp\left(-\frac{\zeta(2)}{(2\pi)^2} s_2 t^2 + \frac{\zeta(4)}{2(2\pi)^4} s_4 t^4 - \dots + \frac{(-1)^k \zeta(2k)}{k(2\pi)^{2k}} s_{2k} t^{2k} \pm \dots\right) \\ &\equiv \exp\left(-\tau_1 t^2 + \frac{1}{2}\tau_2 t^4 - \dots + (-1)^k \frac{1}{k}\tau_k t^{2k} \pm \dots\right)\end{aligned}\quad (41)$$

I have introduced Sylvester's notation,

$$\tau_k = \frac{\zeta(2k)}{(2\pi)^{2k}} s_{2k} = \frac{(-1)^{k+1}}{2(2k)!} (-2k)\zeta(-(2k-1)) s_{2k} = \frac{(-1)^{k+1}}{2(2k)!} B_{2k} s_{2k} \quad (42)$$

on the modern definition of the Bernoulli numbers. In some ways it is better to leave these, and similar, constants in terms of ζ -function values.

The final step in determining the multiplicative sequence is to expand the right-hand side of (41) as a power series in t^2 . This can be done elegantly by relating this to the generating function of homogeneous products which is, (*e.g.* Littlewood, [30]),

$$1 - h_1 x + h_2 x^2 - \dots = \prod_i \frac{1}{1 + x_i x}$$

where h_r is the sum of all homogeneous products of degree r of the indeterminates, x_i . Taking logs, expanding that on the right-hand side, performing the sum on i and then exponentiating, to get back to the left-hand side, gives

$$1 - h_1 x + h_2 x^2 - \dots = \exp\left(-s_1 x + \frac{1}{2}s_2 x^2 - \dots\right) \quad (43)$$

and the h_r are to be considered as functions of the sums of powers, s_q , of the x_i . Comparison with (41) shows that

$$Q_1 Q_2 \dots = 1 - H_1 t^2 + H_2 t^4 - H_3 t^6 + \dots + (-1)^r H_r t^{2r} + \dots \quad (44)$$

where the H_r are the same functions of the τ_i as the h_r are of the s_i . (A separate symbol is therefore not really needed.)

Solving the recursions that follow from the logarithmic derivative of (43) gives Brioschi's determinantal expression for the H_r , (Faà de Bruno, [31] §89, [30]),

$$H_r = \frac{1}{r!} \begin{vmatrix} \tau_1 & -1 & 0 & 0 & 0 \\ \tau_2 & \tau_1 & -2 & 0 & 0 \\ \tau_3 & \tau_2 & \tau_1 & -3 & 0 \\ \dots & & & & \\ \tau_r & \tau_{r-1} & \dots & & \end{vmatrix}. \quad (45)$$

Sylvester's final answer is,

$$W_1 = \frac{1}{\prod_i d_i} \left(\frac{\bar{l}^{d-1}}{(d-1)!} H_0 - \frac{\bar{l}^{d-3}}{(d-3)!} H_1 + \frac{\bar{l}^{d-5}}{(d-5)!} H_2 + \dots \right), \quad (46)$$

which is the same as (33) expressed slightly differently ($H_0 = 1$). I observe that Sylvester was led naturally to multiplicative sequences.

Sylvester, [27], lists six explicit (numerical) polynomials, which are also given by Ehrhart, [8].

6. The second wave

There is a wave for every root of unity, I discuss in this paper only that one corresponding to the root -1 , denoted by W_2 , which, like W_1 , can be algebraically expressed. Sylvester, [21], provides the following algebraic statement: W_2 equals the coefficient of $1/t$ in the generating function,

$$\frac{(-1)^l e^{lt}}{\prod_i (1 - e^{-\alpha_i t}) \prod_j (1 + e^{-\beta_j t})} \quad (47)$$

where the α_i are the even degrees, and the β_j are the odd ones.

In order to express this in a form similar to (27), I recall the expansion formula for the generalised Eulerian functions $E_\nu^{(n)}$, *e.g.* [24,28],

$$\frac{2^n e^{xt}}{\prod_{j=1}^n (e^{\beta_j t} + 1)} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} E_\nu^{(n)}(x \mid \beta)$$

to which I add, for handiness, the one for Bernoulli functions, already used in (27),

$$\frac{t^n \prod_i \alpha_i e^{xt}}{\prod_{i=1}^n (e^{\alpha_i t} - 1)} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} B_\nu^{(n)}(x \mid \alpha).$$

Constructing the product in (47) the condition for the removal of the exponential is

$$\bar{l} = x + y - \frac{1}{2} \sum_{i=1}^{\alpha} \alpha_i - \frac{1}{2} \sum_{j=1}^{\beta} \beta_j, \quad (\alpha + \beta = d), \quad (48)$$

which allows two simplest (in terms of \bar{l}) solutions,

$$x = \bar{l} + \frac{1}{2} \sum \alpha_i \quad y = \frac{1}{2} \sum \beta_j$$

and

$$x = \frac{1}{2} \sum \alpha_i \quad y = \bar{l} + \frac{1}{2} \sum \beta_j,$$

the first of which yields the relevant coefficient required by Sylvester's rule as,

$$W_2 = \frac{(-1)^l}{2^{\beta}(\alpha-1)! \prod \alpha_i} \sum_{\nu=0}^{\alpha-1} \binom{\alpha-1}{\nu} B_{\nu}^{(\alpha)}(\bar{l} + \sum \alpha_i/2 \mid \alpha) E_{\alpha-1-\nu}^{(\beta)}[\beta], \quad (49)$$

while the second form leads to,

$$W_2 = \frac{(-1)^l}{2^{d-1}(\alpha-1)! \prod \alpha_i} \sum_{\nu=0}^{\alpha-1} \binom{\alpha-1}{\nu} 2^{\nu} D_{\alpha-\nu-1}^{(\alpha)}[\alpha] E_{\nu}^{(\beta)}(\bar{l} + \sum \beta_j/2 \mid \beta). \quad (50)$$

The constants $E_{\nu}^{(n)}[\beta] \equiv E_{\nu}^{(n)}(\sum \beta_i/2 \mid \beta)$ are zero for odd ν , [24].

Either form shows that W_2 is a 2-periodic function of l , through $(-1)^l$, multiplied by a polynomial in \bar{l} . For an odd (even) number, α , of even degrees, the polynomial has even (odd) order. If there is only one even degree, this polynomial is a constant, while W_2 is zero if all degrees are odd. (No degree has a factor of two.)

It is possible to check reciprocity for W_2 directly. Changing the sign of \bar{l} in the summand gives a factor of $(-1)^{\nu}$ which, because of the restriction on the index of the E , equals $(-1)^{\alpha-1}$ which can be taken outside the sum. Also the sign $(-1)^l$ changes to $(-1)^{-l-\sum \alpha_i-\sum \beta_j} = (-1)^{l-\sum \beta_j} = (-1)^{l-\beta}$ so that the total change is to give a factor of $(-1)^{l-\beta-\alpha+1} = (-1)^{l-d+1}$ in place of $(-1)^l$, which is the required reciprocity.

Just for comparison, I give the formula that follows from an ' l -solution',

$$x = l + \sum \alpha_i \quad y = \sum \beta_j$$

of (48), viz,

$$W_2 = \frac{(-1)^l}{2^{\beta}(\alpha-1)! \prod \alpha_i} \sum_{\nu=0}^{\alpha-1} \binom{\alpha-1}{\nu} B_{\nu}^{(\alpha)}(l + \sum \alpha_i \mid \alpha) E_{\alpha-1-\nu}^{(\beta)}(\sum \beta_j \mid \beta). \quad (51)$$

This form is given in the interesting paper by Rubinstein and Fel, [22]. It is, perhaps, less natural, reciprocity being harder to spot. Rubinstein and Fel also write the other waves in a similar way more complicatedly using the generalised Eulerian polynomials defined by Carlitz, [32].

I now derive a much neater expression for W_2 . As might be expected, Sylvester provides a route to an explicit solution, which I describe more compactly. I therefore begin again and am required to compute the coefficient,

$$W_2 = (-1)^l \text{co}_{-1} \frac{e^{\bar{l}t}}{\prod_i 2 \sinh \frac{1}{2} \alpha_i t \prod_j 2 \cosh \frac{1}{2} \beta_j t}.$$

Similar to the W_1 computation, expansion of the numerator gives the coefficient of the power $(-1)^l \bar{l}^n / n!$ as,

$$\begin{aligned} & \text{co}_{-1} \frac{t^n}{\prod_i 2 \sinh \frac{1}{2} t \alpha_i \prod_j 2 \cosh \frac{1}{2} t \beta_j} \\ &= \frac{1}{2^\beta \prod \alpha_i} \text{co}_{\alpha-1-n} \prod_i \frac{\xi_i}{\sinh \xi_i} \prod_j \frac{1}{\cosh \eta_j}, \end{aligned} \quad (52)$$

where $\xi_i = t \alpha_i / 2$ and $\eta_j = t \beta_j / 2$. The product of two multiplicative series now arises. One has been encountered previously, see (37), (38), and I refer to it as the untwisted series. I now cover the other, defined by the characteristic function,

$$\Omega(y) = \frac{1}{\cosh y} = \prod_{n=1}^{\infty} \left(1 + \frac{y^2}{((n-1/2)\pi)^2} \right)^{-1}, \quad (53)$$

in like manner⁶, and call this the twisted series. Thus,

$$\log \Omega(y) = -\frac{\zeta_R(2, 1/2)}{\pi^2} y^2 + \frac{\zeta_R(4, 1/2)}{2\pi^4} y^4 - \dots + (-1)^k \frac{\zeta_R(2k, 1/2)}{k \pi^{2k}} y^{2k} \pm \dots,$$

in terms of the Riemann-Hurwitz ζ -function, $\zeta_R(s, w)$.

The similarity of this to (40) shows that the formal algebra is just the same, only the numerical coefficients differ, with B_k replaced by $(2^{2k} - 1)B_k$. So I define,

$$\varsigma_k = \frac{\zeta_R(2k, 1/2)}{(2\pi)^{2k}} s_{2k} = \frac{(-1)^{k+1} (2^{2k} - 1)}{2(2k)!} B_{2k} s_{2k}, \quad (54)$$

⁶ The products (53) and (39) can be considered as examples of the Mittag-Leffler theorem which notion allows one to generalise the approach.

and the multiplicative sequence this time is (*cf* (44)),

$$\Omega_1 \Omega_2 \dots = 1 - H_1(\varsigma) t^2 + H_2(\varsigma) t^4 - H_3(\varsigma) t^6 + \dots + (-1)^r H_r(\varsigma) t^{2r} + \dots \quad (55)$$

According to (52) a product of multiplicative sequences is required. This can be done from the series (44) and (55) but is best performed first at the exponential level using (41) and,

$$\Omega_1 \Omega_2 \dots = \exp \left(-\varsigma_1 t^2 + \frac{1}{2} \varsigma_2 t^4 - \dots + (-1)^k \frac{1}{k} \varsigma_k t^{2k} \pm \dots \right). \quad (56)$$

Doing this, and then expanding I find, as before,

$$Q_1 Q_2 \dots \Omega_1 \Omega_2 \dots = 1 - H_1(\varsigma + \tau) t^2 + H_2(\varsigma + \tau) t^4 - \dots + (-1)^r H_r(\varsigma + \tau) t^{2r} + \dots \quad (57)$$

The final answer for the second wave is then formally similar to that for the first as a terminating series,

$$W_2 = (-1)^l \frac{1}{2^\beta \prod_j \alpha_i} \left(\frac{\bar{l}^{\alpha-1}}{(\alpha-1)!} H_0 - \frac{\bar{l}^{\alpha-3}}{(\alpha-3)!} H_1 + \frac{\bar{l}^{\alpha-5}}{(\alpha-5)!} H_2 + \dots \right), \quad (58)$$

where $H_i \equiv H_i(\tau + \varsigma)$. τ_i defined by (42) with s_{2q} the sums of powers of the even degrees and ς_i is defined by (54) with, this time, s_{2q} being the sums of powers of the odd degrees. I could not find this exact form in Sylvester's writings.

From a formal point of view, I note that the functions associated with the twisted multiple sequence are the generalised Euler polynomials.

From a *spectral* perspective, the expansion variable is the eigenvalue, ω , rather than l , hence I require

$$(-1)^{d-1} B_{d-1}^{(d)}(a - \omega \mid \mathbf{d}) = (-1)^{d-1} \sum_{s=0}^{d-1} \binom{d-1}{s} (-\omega)^s B_{d-1-s}^{(d)}(a \mid \mathbf{d}). \quad (59)$$

So that the polynomial, P , is

$$P = \sum_{k=0}^{d-1} P_k \omega^k$$

with

$$\begin{aligned} P_k &= \frac{(-1)^{d-1-k}}{(d-1-k)! k! \prod_i d_i} B_{d-1-k}^{(d)}(a \mid \mathbf{d}) \\ &= \frac{2}{\Gamma((k+1)/2)} C_{(d-k-1)/2} \end{aligned}$$

in terms of the heat-kernel coefficients, C_* , on S^d/Γ computed in [7].

7. Ehrhart quasipolynomials

Regarding the Ehrhart polynomial, I note that this is what could be termed an ‘accumulated degeneracy’, defined by,

$$G(l) = \sum_{l'=0}^l g(l'), \quad (60)$$

generically expressed. It is commonplace in generating function circles to write the obvious recursion for G as

$$H(\sigma) = \frac{1}{1-\sigma} h(\sigma) \quad (61)$$

for $H(\sigma) \equiv \sum_{l=0}^{\infty} G(l) \sigma^l$ etc.

Exactly the present situation of spectra on regular tessellations of the d -sphere has already been considered in [33] (extended to p -forms) and some discussion of the asymptotics were there made, using generating functions.

Looking back at say (13), the recursion division in (61) is tantamount to adding 1 to list of degrees, d_i , and increasing d by one.⁷ The denominator for this case is the Ehrhart quasipolynomial $L(\overline{\mathcal{P}}_{\mathbf{d}}, l)$, *i.e.*

$$L(\overline{\mathcal{P}}_{\mathbf{d}}, l) = \frac{l}{\mathbf{d}, 1}.$$

The strictly polynomial part, P , of the Ehrhart quasipolynomial is given by the coefficient of $1/t$ in,

$$\frac{e^{lt}}{\prod_{i=1}^d (1 - e^{-d_i t})(1 - e^{-t})},$$

which has already been found in (27) as (with $a = (d-1)/2$),

$$\begin{aligned} P &= \frac{(-1)^d}{d! \prod_i d_i} B_d^{(d+1)}(a - \omega \mid \mathbf{d}, 1) \\ &= \frac{(-1)^d}{d! \prod_i d_i} \sum_{k=0}^d \binom{d}{k} (-\omega)^k B_{d-k}^{(d+1)}(a \mid \mathbf{d}, 1). \end{aligned} \quad (62)$$

The relation with Todd polynomials gives

$$B_r^{(d+1)}(x \mid \mathbf{d}, 1) = (-1)^r r! \sum_{s=0}^r (-1)^s \frac{x^s}{s!} T_{r-s}(\overline{\sigma}_1, \dots, \overline{\sigma}_{r-s}), \quad r \leq d, \quad (63)$$

⁷ This is seen by noting that adding $1m_3$ to the left-hand side of (12) is equivalent to adding the results for (12) with the right-hand side equalling $l, l-1, l-2, \dots, 0$ in turn. See [27], [8], [2].

where now the $\overline{\sigma}_s$ are symmetric functions of the degrees d_i ($i = 1, \dots, d$) and 1.

First, I wish to express the symmetric functions, $\overline{\sigma}_s$, of d_i , $i = 1, \dots, d$ and 1 in terms those, σ_s , of just the d_i since these are the variables used by [2], although this is not essential. Trivially, or from the defining fundamental identities,

$$\prod_{i=1}^d (1 + d_i t) = \sum_0^d \sigma_r t^r$$

$$\prod_{i=1}^d (1 + d_i t)(1 + t) = \sum_0^d \overline{\sigma}_r t^r,$$

it follows that

$$\overline{\sigma}_s = \sigma_s + \sigma_{s-1}. \quad (64)$$

For convenience I list a few resulting Todd polynomials obtained from the usual ones,

$$\overline{T}_0 = 1, \quad \overline{T}_1 = \frac{1}{2}(\sigma_1 + 1), \quad \overline{T}_2 = \frac{1}{12}(\sigma_1^2 + \sigma_2 + 3\sigma_1 + 1) \quad (65)$$

and write down some Bernoulli functions from (28),

$$B_1^{(d+1)}((d-1)/2 \mid \mathbf{d}, 1) = \frac{d-1}{2} \overline{T}_0 - \overline{T}_1$$

$$B_2^{(d+1)}((d-1)/2 \mid \mathbf{d}, 1) = 2\overline{T}_2 - (d-1)\overline{T}_1 + \left(\frac{d-1}{2}\right)^2 \overline{T}_0$$

The first two members of the polynomial (62) now read

$$P = \frac{(-1)^d}{d! \prod_i d_i} \left(\omega^d + \frac{d(2 + \sigma_1 - d)}{2} \omega^{d-1} + \dots \right)$$

$$= \frac{(-1)^d}{d! \prod_i d_i} \left(l^d + \frac{d(\sigma_1 + 1)}{2} l^{d-1} + \dots \right) \quad (66)$$

where, for the combinatorialists, and as a check, I have transformed the series to one in l . These forms should be compared with (5) and (11).

Of course it is not necessary to go via ω to l , one can write (no different to (62)),

$$P = \frac{(-1)^d}{d! \prod_i d_i} B_d^{(d+1)}(l \mid \mathbf{d}, 1)$$

$$= \frac{(-1)^d}{d! \prod_i d_i} \sum_{k=0}^d \binom{d}{k} (-l)^k B_{d-k}^{(d+1)}(\mathbf{d}, 1).$$

$$= \frac{1}{\prod_i d_i} \sum_{k=0}^d \frac{l^k}{k!} T_{d-k}(\overline{\sigma}_1, \dots, \overline{\sigma}_{d-k}) \quad (67)$$

which is really only a re-expression. Using the forms of the Todd polynomials, (65), published coefficients are readily confirmed in rapid fashion.

However, rather than use these expressions, it is far easier to take advantage of Sylvester's explicit expansion, (46) and, making the adjustments to move to the Ehrhart polynomial, I get for its polynomial part,

$$P = \frac{1}{\prod_i d_i} \left(\frac{\bar{l}^d}{d!} H_0 - \frac{\bar{l}^{d-2}}{(d-1)!} H_1 + \frac{\bar{l}^{d-4}}{(d-2)!} H_2 + \dots \right), \quad (68)$$

where the H_i are the same functions as before, (45), but of *new* τ s defined by,

$$\tau_k = \frac{\zeta(2k)}{(2\pi)^{2k}} (s_{2k} + 1) = \frac{(-1)^{k+1} B_{2k}}{2(2k)!} (s_{2k} + 1), \quad (69)$$

obtained by adding 1 to the list of degrees. The s_q are sums of powers of the d_i .

Particular cases are easily constructed and the general one machine coded.

I can now return to the recipe given in section 2 and construct the combination of Ehrhart polynomials,

$$\frac{1}{2} [A(\omega) + A(\omega - 1)] = \frac{(-1)^d}{d! \prod_i d_i} \left(\omega^d + \frac{d(\sigma_1 - d + 1)}{2} \omega^{d-1} + \dots \right)$$

which agrees with the first two terms of the asymptotic expansion, (9), of the smoothed counting function, since $b_1 = \sigma_1 - d + 1$ and confirms the statement made in section 2, for the first two terms.

8. Conclusion

It has been shown that the first two terms in the orthodox asymptotic expansion of the smoothed counting function can be obtained from a symmetrical combination of two Ehrhart polynomials in any dimension.

It is, perhaps, no surprise that the heat-kernel coefficients can be obtained from the Ehrhart polynomial as this encodes all the eigenvalue information.

The technical evaluation of denominators is greatly eased by the use of multiplicative sequences. Indeed these arise naturally. A simple expression for the second wave involving the numbers of all homogeneous products. In a later paper I intend to look at waves beyond the second one.

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